The principle of maximum randomness in the theory of developed turbulence: A model with maximum parity violation

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Theory of turbulence

General features

Main object is incompressible velocity field **v**, $\partial_i v_i = 0$

Navier-Stokes equation.

$$\partial_t v_i = -v_j \partial_j v_i + \nu \Delta v_i - \frac{1}{\rho} \partial_i P$$

Laminar flow

- Easy
- Well-studied
- Possible to obtain precise solutions of N-S equations

Turbulent flow

- Purely stochastic
- No precise solution could be obtained
- First statistical theory is due to Kolmogorov, 1940s.

Kolmogorov's formalism

The 5/3 law

One-dimensional longitudinal spectral density

$$< v_1^2 >= \int_0^\infty E_1(k)dk$$

The following holds:

$$E_1(k) = C_k W^{2/3} k^{-\frac{5}{3}},$$

W - mean energy spectral flux, C_k - Kolmogorov constant

Notes

- Follows from dimensional analysis
- Good agreement with experiments
- Impossible to determine value of C_k
- Small shift from Kolmogorov scaling is possible (Kolmogorov, 1962; modern RG)

Model

Navier-Stokes equation (spectral energy balance equation)

$$\partial_t < E_{ij}(\mathbf{k}) > /2 =< \dot{E}_{ij}(\mathbf{k}) > + d_{ij}(\mathbf{k}) = 0$$

Where

$$E_{ij}(\mathbf{k}) = v_i(\mathbf{k}) v_j(-\mathbf{k})$$

$$\dot{E}_{ij}(\mathbf{k}) = -\nu k^2 E_{ij}(\mathbf{k}) - T_{ij}(\mathbf{k})$$

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Applying maximal randomness ideas:

Distribution function

$$\sigma = - < \ln \rho >= - \int D\mathbf{v} \rho(\mathbf{v}) \ln \rho(\mathbf{v})$$

$$\rho(\mathbf{v}) = Q^{-1} exp[2\pi^{-3} \int d\mathbf{k} \lambda_{ij}(\mathbf{k}) \dot{E}_{ij}(\mathbf{k})]$$

What are we trying to get

Correlation function

$$G_{ij}^{vv}(\mathbf{k}) = < v_i(\mathbf{k}) v_j(-\mathbf{k}) >$$

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Applying symmetry and following Kolmogorov's ideas, one may assume (in the inertial interval):

$$G_{ij}^{vv}(\mathbf{k}) = C \cdot P_{ij}(\mathbf{k}) \cdot k^{-\gamma}$$

Kolmogorov's theory predicts value of γ equals 11/3, which is in good agreement with experimental data.

Diagram technique – I

After using standard procedures: MSR-formalism and Dyson equations, we get set of self-consistent equations for correlation functions.

Equations

$$(G^{vv})^{-1} = \varphi_0 k^2 - \Sigma^{vv} - \tilde{\lambda}^2 \Sigma^{uu} - \tilde{\lambda}(\Sigma^{uv} + \Sigma^{vu}),$$

$$G^{vv}(\Sigma^{uv} + \tilde{\lambda}\Sigma^{uu}) = 0.$$

Result of calculations

- Diagrams are free from IR-divergencies in all loops of perturbation theory
- Kolmogorov indices
- Negative Kolmogorov constant

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The idea is to consider parity violation of velocity field.

Parity violation

Symmetries

- Translation symmetry
- Rotational symmetry
- P-symmetry

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- δ_{ij}
- $k_i k_j$
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New expression for G

$$G_{ij}^{vv}(\mathbf{k}) = C \cdot k^{-\gamma}(P_{ij}(\mathbf{k}) + i\theta\varepsilon_{ijl}k_l/k)$$

Diagram technique – II

Obtaining the same equations for model with the violation of parity.

Result of calculations in 1-loop approximation

- Diagrams are free from IR-divergencies in all loops of perturbation theory
- Kolmogorov indices
- Positive Kolmogorov constant
- Parameter $|\theta|$ equals 1.03, while the allowed range is $|\theta| \leq 1$

Diagram technique – II

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Result of calculations in 1-loop approximation

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Global analysis of perturbation theory serie

- Almost all UV-divergencies could be removed by means of the renormalization procedure
- 2 loops contain logarithmic divergencies
- It may lead to anomalous scaling, i.e. small shift from Kolmogorov's indice $g = 11/3$

Two-loop calculations

$$G^{vv}_{ij}(\Sigma^{uv}_{js} + \tilde{\lambda}_{jk}\Sigma^{uu}_{ks}) = 0$$

Where G is correlator and $\tilde{\lambda}$ is Lagrange multiplier:

$$G^{vv}_{ij} = bk^{-2\gamma}(P_{ij} + i\theta\varepsilon_{ijs}k_s/k)$$

$$\tilde{\lambda}_{ij} = ak^{2\sigma}(P_{ij} + i\chi\varepsilon_{ijs}k_s/k)$$

2-loop logarithmically UV-divergent diagram:

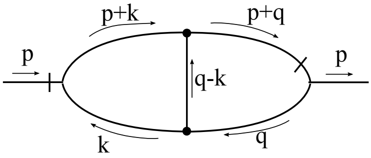

Figure :

Programming part

form

1. Initializing expression
2. Opening brackets, integrating, subtracting tensors
3. Run unit tests, check the answer
4. 10^8 summands in 'bottle neck'
5. huge analytical scalar answer (written via recursive functions)

regexp

Text processing (form $\to$ Maple)

Maple

1. Working with answer as mathematical expression
2. Rewriting recursive functions as products of Γ-functions
3. Obtaining numerical answers

Inverse propagator

Inverse propagator – no PV

$$G^{vv}_{ij} = bk^{-2\gamma} P_{ij}$$

$$(G^{vv}_{ij})^{-1} = b^{-1} k^{2\gamma} P_{ij}$$

Inverse propagator – PV

$$G^{vv}_{ij} = bk^{-2\gamma} (P_{ij} + \theta Q_{ij})$$

$$(G^{vv}_{ij})^{-1} = \frac{b^{-1} k^{2\gamma}}{1 - \theta^2} (P_{ij} - \theta Q_{ij})$$

So, $\theta \sim 1$ does imply strong instability.

Model with maximum parity violation

We choose $\theta = \chi = 1$, so

$$G^{vv}_{ij} = bk^{-2\gamma}(P_{ij} + i\varepsilon_{ijs}k_s/k)$$

New unit tensor

$$U_{ij} = (P_{ij} + i\varepsilon_{ijs}k_s/k)/2$$

Inverse propagator

$$(G^{vv})^{-1}_{ij} = \frac{b^{-1}k^{2\gamma}}{4}(P_{ij} + i\varepsilon_{ijs}k_s/k),$$

And the pole and instability over θ are gone.

Applying the model

A closed system of equation on parameters a, b, ε.

$$(G^{vv})^{-1} = \varphi_0 k^2 - \Sigma^{vv} - \tilde{\lambda}^2 \Sigma^{uu} - \tilde{\lambda}(\Sigma^{uv} + \Sigma^{vu})$$
$$(G^{vv})^{-1} = \Sigma^I + \Sigma^{II} + \ldots$$

We've calculated first two summand in all series.

properties

- Second summand is much greater than first
- Series lie beyond convergent region
- No straight way to determine actual growth rate

We assume series growth to be exponential

Final values

$$\varepsilon = 0.0894$$
$$C_k = 1.172$$

Overall results

- Model with a maximum violation of parity, similar to considered in the theory of weak interactions, is presented
- Ultraviolet divergences of 2-loop diagrams of the model were analyzed
- The existence of anomalous scaling (small shift from Kolmogorov index) is demonstrated
- The values of anomalous scaling and Kolmogorov constant are obtained.

The end

Thank you!